10

Applications to the Study of Differential Equations

Canonical transforms, besides generalizing the Fourier and Bargmann integral transforms, provide a fine tool for the analysis of a class of differential equations. The class consists of up-to-second-order differential operators of *parabolic* type. These include the diffusion, the Schrödinger free-particle, the linear potential (free-fall), and the attractive and repulsive oscillator equations. It also includes a few others such as the Fokker–Planck equation. Although this class is far from universal, the ease with which solutions and properties are found makes canonical transforms an attractive tool for problems such as these. In Section 10.1 we start with the introduction of *inhomogeneous* linear canonical transformations and apply the machinery to a deeper study of the diffusion equation: how to find families of solutions out of a known solution (the action of the *similarity group* of the equation) and the question of *separating coordinates*, which brings us to generalized normal modes. In Section 10.2 the analysis is applied to a general member of the differential equation class. We show that all computations reduce to, essentially, 2×2 matrix algebra. This is in the true spirit of group theory.

10.1. The Diffusion Equation: Similarity Group and Separating Coordinates

We shall consider here the set of all up-to-second-order operators in $\mathbb{Q}$ and $\mathbb{P}$,

$$\mathbb{H} = A\mathbb{P}^2 + B(\mathbb{Q}\mathbb{P} + \mathbb{P}\mathbb{Q}) + C\mathbb{Q}^2 + D\mathbb{Q} + E\mathbb{P} + F\mathbb{1}, \qquad A, B, \ldots, F \in \mathscr{C}, \quad (10.1)$$

and shall introduce *inhomogeneous* canonical transforms, i.e., the group of canonical transforms of Chapter 9 plus translations and multiplications by $c \exp(\alpha q)$. Operators (10.1) appear in a class of *parabolic* differential equations

$$\mathbb{H}f(q, t) = -i \frac{\partial}{\partial t} f(q, t), \tag{10.2}$$

for which we can find "normal modes" and separating variables. We shall show that the mathematical techniques we have developed reduce all the needed computations to essentially 2×2 matrix algebra. In this section we shall apply all developments to the *diffusion equation*, where $\mathbb{H}$ in (10.1) is simply $-i\, \partial^2/\partial q^2 = i\mathbb{P}^2$. The slight complication of having complex parameters will be more than offset by the readily interpretable results. Section 10.2 will show how the general case is to be handled.

10.1.1. Heisenberg–Weyl Transformations

The added generality of (10.1)–(10.2) over the corresponding equations we considered in Chapter 9 [Eqs. (9.76) and (9.81)] consists of allowing for terms linear in $\mathbb{Q}$, $\mathbb{P}$, and $\mathbb{1}$. To implement the ideas of Chapter 9 for this enlarged set, we must examine their exponentiated action on functions and operators. Most of this has been done in Chapter 7, where we saw that $\mathbb{P}$ generates *translations* of the space [Eqs. (7.27) and (7.69)], $\mathbb{Q}$ generates multiplication of the function by an exponential [Eq. (7.70)], and $F\mathbb{1}$, quite simply, multiplies the function by a constant. From these we can define the $\mathbb{W}$ *transform* operators

$$\begin{aligned}\mathbb{W}(x, y, z) &:= \exp[i(x\mathbb{Q} + y\mathbb{P} + z\mathbb{1})] \\ &= \exp[i(z + xy/2)] \exp(ix\mathbb{Q}) \exp(iy\mathbb{P}) \\ &= \exp[i(z - xy/2)] \exp(iy\mathbb{P}) \exp(ix\mathbb{Q}), \qquad x, y, z \in \mathscr{R}. \end{aligned} \tag{10.3a}$$

The equality between the last two expressions can be proven by applying the operators to any analytic function $f(q)$, obtaining in both cases

$$\big(\mathbb{W}(x, y, z)\mathbf{f}\big)(q) = \exp[i(xq + xy/2 + z)]f(q + y) \tag{10.3b}$$

[recall the Weyl commutation relation (7.33)]. The equality with the first form of $\mathbb{W}$ takes slightly longer to verify (see Exercise 10.1). Moreover, the set of all operators (10.3) for $x, y, z \in \mathscr{R}$ gives rise to a *group* of transformations, which we shall denote by W. (a) The *composition* of two elements of the set (10.3) is a new operator which is again a member of the set. Indeed,

$$\begin{aligned}&\mathbb{W}(x_2, y_2, z_2)\mathbb{W}(x_1, y_1, z_1) \\ &\qquad = \mathbb{W}\big(x_2 + x_1, y_2 + y_1, z_2 + z_1 + (y_2x_1 - x_2y_1)/2\big),\end{aligned} \tag{10.4}$$

as can easily be verified by acting with both members on any $f(q)$ and using (10.3b). Equation (10.4) shows that the product of two $\mathbb{W}$ transforms is a $\mathbb{W}$ transform. (b) *Associativity* obviously holds. (c) The identity of the group is $\mathbb{W}(0, 0, 0) = \mathbb{1}$. (d) $[\mathbb{W}(x, y, z)]^{-1} = \mathbb{W}(-x, -y, -z)$, as can be verified from (10.4).

Exercise 10.1. Show that the first equality in (10.3a) holds. This can be seen if we develop both members in series and compare coefficients of like powers of x, y, and z inductively.

A simpler proof is obtained if you use (9.38) in order to write the identity

$$\mathbb{C}_M \exp(i\mathbb{P})\mathbb{C}_{M^{-1}} = \exp[i(x\mathbb{Q} + y\mathbb{P})] \qquad \text{for } \mathbf{M} = \begin{pmatrix} y & 0 \\ -x & y^{-1} \end{pmatrix} \tag{10.5}$$

and then use (9.23), (7.27), and (7.69). The action of (10.5) on any $f(q)$ is given by (10.3b) for $z = 0$. [The parameter z is rather trivial to extract from the Baker–Campbell–Hausdorff relation in (10.3a) as $\mathbb{1}$ commutes with $\mathbb{Q}$ and $\mathbb{P}$.]

Exercise 10.2. Show that for x, y, and z *real* the set of $\mathbb{W}$ transforms is a group of *unitary* mappings of $\mathscr{L}^2(\mathscr{R})$ onto itself. Further description of the *Heisenberg–Weyl* group (10.3b) can be found in Wolf and García (1972) [see also Wolf (1975)].

10.1.2. Inhomogeneous Linear Transformations

The action of the elements of the Heisenberg–Weyl group W on *operators* can be ascertained to be

$$\begin{aligned} \mathbb{W}(D\mathbb{Q} + E\mathbb{P})\mathbb{W}^{-1} &= D\mathbb{W}\mathbb{Q}\mathbb{W}^{-1} + E\mathbb{W}\mathbb{P}\mathbb{W}^{-1} \\ &= D(\mathbb{Q} + y\mathbb{1}) + E(\mathbb{P} - x\mathbb{1}) \\ &= D\mathbb{Q} + E\mathbb{P} + (Dy - Ex)\mathbb{1}. \end{aligned} \tag{10.6}$$

The proof can rely on letting the members act on $\mathscr{C}^\infty$ functions $f(q)$ as with (10.3b) or can use only the linear operators $\mathbb{Q}$ and $\mathbb{P}$ and their properties: (a) linearity; (b) the commutator of $\mathbb{Q}$ and $\mathbb{P}$ is $i\mathbb{1}$ [Eq. (7.59b)]; (c) the set of operators $\mathscr{S}_1 := \{D\mathbb{Q} + E\mathbb{P} + F\mathbb{1};\, D, E, F \in \mathscr{C}\}$ closes on itself under commutation (hence forms a *Lie algebra*) and when exponentiated, generates the group W; (d) the formula (9.78). The group of transformations (10.3) thus acts on the set $\mathscr{S}_1$ as if it were a three-dimensional space with Cartesian coordinates D, E, F, although, note, the parameter z in (10.3) does not appear in (10.6). This three-dimensional space of operators $\mathscr{S}_1$ is thus acted upon by W in addition to being acted upon by the group $SL(2, \mathscr{R})$ of linear canonical transforms which occupied Chapter 9 [Eqs. (9.1)]. The two groups can then be composed [as a *semidirect product* of $SL(2, \mathscr{R})$ and W] and the elements of the product denoted by

$$\mathbb{I}\{\mathbf{M}, \boldsymbol{\xi}, z\} = \mathbb{I}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x, y, z)\right\} := \mathbb{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbb{W}(x, y, z), \quad (10.7a)$$

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \qquad \boldsymbol{\xi} := (x, y). \quad (10.7b)$$

The set I of transformations (10.7) also forms a group. (a) The product of two of its elements is again an element of the set:

$$\begin{aligned} \mathbb{I}\{\mathbf{M}_2, \boldsymbol{\xi}_2, z_2\} \cdot \mathbb{I}\{\mathbf{M}_1, \boldsymbol{\xi}_1, z_1\} &= \mathbb{C}(\mathbf{M}_2)\mathbb{W}(\boldsymbol{\xi}_2, z_2)\mathbb{C}(\mathbf{M}_1)\mathbb{W}(\boldsymbol{\xi}_1, z_1) \\ &= \mathbb{C}(\mathbf{M}_2)\mathbb{C}(\mathbf{M}_1)\mathbb{C}(\mathbf{M}_1^{-1})\mathbb{W}(\boldsymbol{\xi}_2, z_2)\mathbb{C}(\mathbf{M}_1^{-1})^{-1}\mathbb{W}(\boldsymbol{\xi}_1, z_1) \\ &= \mathbb{C}(\mathbf{M}_2\mathbf{M}_1)\mathbb{W}(\boldsymbol{\xi}_2\mathbf{M}_1, z_2)\mathbb{W}(\boldsymbol{\xi}_1, z_1) \\ &= \mathbb{I}\{\mathbf{M}_2\mathbf{M}_1, \boldsymbol{\xi}_2\mathbf{M}_1 + \boldsymbol{\xi}_1, z_2 + z_1 + \tfrac{1}{2}\boldsymbol{\xi}_2\mathbf{M}_1\boldsymbol{\Omega}\boldsymbol{\xi}_1{}^T\}, \\ \boldsymbol{\Omega} &:= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (10.8)$$

The statement is proven in the next to last member. The last equality is again the definition (10.7) written so as to bring out the fact that the product of elements of W in (10.4) can be abbreviated using vector notation: $\boldsymbol{\xi}$ as in (10.7b) and $\boldsymbol{\xi}^T$ as the column vector transpose to $\boldsymbol{\xi}$. (b) Associativity holds for I as it does for its constituents. (c) The *unit* element for the set is $\mathbb{I}\{\mathbf{1}, \mathbf{0}, 0\}$. (d) The *inverse* of any operator (10.7) is $\mathbb{I}\{\mathbf{M}, \boldsymbol{\xi}, z\}^{-1} = \mathbb{I}\{\mathbf{M}^{-1}, -\boldsymbol{\xi}\mathbf{M}^{-1}, -z\}$, as can easily be verified from (10.8). We shall call I the group of *inhomogeneous linear transformations.*

It can be verified that the subset of I consisting of the $SL(2, \mathscr{R})$ operators $\mathbb{I}\{\mathbf{M}, \mathbf{0}, 0\}$ forms a proper subgroup $SL(2, \mathscr{R}) \subset I$, as does the subset $\mathbb{I}\{\mathbf{1}, x, y, z\}$, which gives $W \subset I$. An important one-parameter subgroup which is not totally contained in $SL(2, \mathscr{R})$ or in W is the following set of operators:

$$\mathbb{I}\left\{\begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}, (\tau, -\tau^2/2, -\tau^3/12)\right\} = \exp[i\tau(\tfrac{1}{2}\mathbb{P}^2 + \mathbb{Q})], \quad (10.9)$$

where the equality still has to be proven.

Exercise 10.3. Show that the elements of I of the form (10.9) do constitute a one-parameter subgroup. If τ_1 and τ_2 are the values of the parameters, their composition will be of the same form with $\tau_1 + \tau_2$. Try disentangling the operator exponential in (10.9) into two such operators following (10.7).

Applying the left-hand side of (10.9) on a function $f(q)$ [as prescribed by (10.7), (10.3b), and the canonical transform (9.5)–(9.8), with $-\tau = \exp(-i\pi)\tau$], we find it is an integral transform with a kernel

$$C_l(q, q') := \exp(3i\pi/4)(2\pi\tau)^{-1/2} \exp\{-i[(q - q')^2/2\tau - (q + q')\tau/2 - \tau^3/24]\}. \tag{10.10}$$

An argument parallel to (9.75) now shows that the one-parameter set (10.9) is indeed generated by the operator $\mathbb{H}^l := \frac{1}{2}\mathbb{P}^2 + \mathbb{Q}$. It is the *linear potential* (free-fall) Schrödinger Hamiltonian, and Eq. (10.10) is the system's Green's function. The eigenfunctions of this operator and its spectrum have been studied in (8.87)–(8.89). Figure 10.1 is a plot of the time evolution of Gaussian initial conditions under the equation (10.2) with $\mathbb{H}^l$ and various values of τ, in analogy to the evolution under $\mathbb{H}^f$, $\mathbb{H}^r$, and $\mathbb{H}^h$ (Figs. 9.3–9.5) with a similar physical interpretation. This *quartet* of operators $\mathbb{H}^\omega$, $\omega = f, r, h, l$, will be quite useful in what follows. In Section 10.2 we shall show that these four are, in fact, *all* the operators we need to consider in connection with the second-order parabolic differential equations (10.1)–(10.2).

10.1.3. Diffusion and Transformation of Initial Conditions

We have assembled most of the mathematical tools we need in order to present the main application in this chapter. The remaining pieces will be

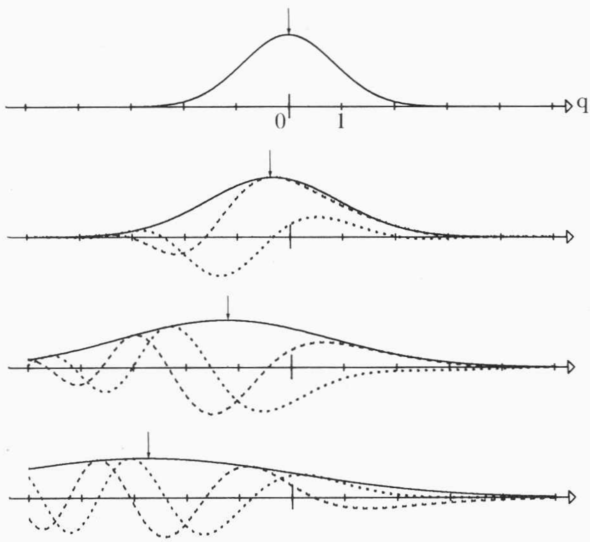


Fig. 10.1. Time development of a Gaussian wave function under the free-fall Schrödinger equation [drawn in the same manner as Figs. 9.3–9.5, following Eq. (9.85) for the evolution operator (10.9)]. The peak of the Gaussian "falls" as a classical particle would.

developed and put in place as we proceed to apply our enlarged set of transforms to answer the following question: Let $f(q)$ be, say, the initial temperature distribution of a thin rod which diffuses in time as $f(q, t)$ subject to

$$\frac{\partial^2}{\partial q^2} f(q, t) = \frac{\partial}{\partial t} f(q, t), \qquad f(q, 0) =: f(q). \tag{10.11}$$

What will be the time development of an *I-transformed* initial condition $(\mathbb{I}\{g\}\mathbf{f})(q)$, where $g = \{\mathbf{M}, \boldsymbol{\xi}, z\} \in I$? The answer turns out to be remarkably simple. As the time development under the diffusion equation is a Gauss–Weierstrass transform (9.67), it follows from (9.81)–(9.82) that

$$f(q, t) = \left[\mathbb{I}\left\{\begin{pmatrix}1 & -2it\\ 0 & 1\end{pmatrix}, (0, 0, 0)\right\}\mathbf{f}\right](q) =: (\mathbb{I}_{H(t)}\mathbf{f})(q). \tag{10.12}$$

The $\mathbb{I}\{g\}$-transformed initial conditions thus give rise to the following temperature distribution:

$$\begin{aligned} f_g(q, t) &:= [\mathbb{I}_{H(t)}\mathbb{I}\{g\}\mathbf{f}](q) \\ &= \left[\mathbb{I}\left\{\begin{pmatrix}1 & -2it\\ 0 & 1\end{pmatrix}, (0, 0, 0)\right\}\mathbb{I}\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}, (x, y, z)\right\}\mathbf{f}\right](q) \\ &= \left[\mathbb{I}\left\{\begin{pmatrix}a - 2ict & b - 2idt\\ c & d\end{pmatrix}, (x, y, z)\right\}\mathbf{f}\right](q) \\ &= \left[\mathbb{I}\left\{\begin{pmatrix}a - 2ict & 0\\ c & (a - 2ict)^{-1}\end{pmatrix}, (x, y + 2it_g x, z)\right\}\right. \\ &\qquad \left.\times\, \mathbb{I}\left\{\begin{pmatrix}1 & -2it_g\\ 0 & 1\end{pmatrix}, (0, 0, 0)\right\}\mathbf{f}\right](q). \end{aligned} \tag{10.13}$$

We have used the fact that both the time-development operator $\mathbb{I}_{H(t)}$ and the applied transformation $\mathbb{I}\{g\}$ are elements of the same transformation group I. This allowed us to compose the two by (10.8) and *decompose* the product in such a way that the time-development operator $\mathbb{I}_{H(t_g)}$ acts first but with a *transformed time variable*:

$$t_g = (dt + ib/2)/(a - 2ict). \tag{10.14}$$

This expression follows from ordinary matrix algebra on the two group elements in the last member according to (10.8). Now $\mathbb{I}_{H(t_g)}$ acting on $f(q)$ will produce a function $f(q, t_g)$ that is identical to (10.11) except for having t_g in place of t. The left-most factor in the last member of (10.13) is a special kind of I transform: as the 1-2 element is zero, it is not an integral transform at all but, from (10.3b) and (9.23), only a *geometric transform*:

$$\begin{aligned} \left[\mathbb{I}\left\{\begin{pmatrix}a & 0\\ c & a^{-1}\end{pmatrix}, (x, y, z)\right\}\mathbf{f}\right](q) &= a^{-1/2}\exp[i(cq^2/2a + xq/a + xy/2 + z)] \\ &\qquad \times f(q/a + y). \end{aligned} \tag{10.15}$$

The geometric transform of a function, we see, involves displacement and change of scale of the argument and multiplication of the function by a Gaussian, an exponential, and a constant factor. Due to the fact that $\mathbb{G}_{H(t)}$ in (10.12) involves an imaginary entry, complex numbers have appeared in (10.13) and (10.14). As the temperature is supposed to be a *real* function of *real* space q and time t, we can redefine for convenience the following group parameters:

$$\beta := ib/2, \qquad \gamma := -2ic, \qquad \xi := -2ix, \qquad \zeta = -iz. \tag{10.16}$$

Writing the entries in (10.13) and (10.14) in terms of these and using the result (10.15), we find that $(\mathbb{G}\{g\}\mathbf{f})(q)$ has developed under diffusion as

$$f_g(q, t) = \mu_g(q, t) f(q_g, t_g), \tag{10.17a}$$

where space q and time t have been transformed to

$$q_g = [q - \xi(\beta + dt)]/(a + \gamma t) + y = q/(a + \gamma t) - \xi t_g + y, \tag{10.17b}$$

$$t_g = (\beta + dt)/(a + \gamma t), \tag{10.17c}$$

and the function f has been factored by a *multiplier* function μ_g with the structure

$$\mu_g(q, t) = C_g(a + \gamma t)^{-1/2} \exp[S(q, t)], \tag{10.17d}$$

$$C_g := \exp[-(\xi y/4 + \zeta)], \tag{10.17e}$$

$$S(q, t) = [-\gamma q^2 - 2\xi q + \xi^2(\beta + dt)]/4(a + \gamma t). \tag{10.17f}$$

Exercise 10.4. Verify these results in all detail.

10.1.4. Manifest and "Hidden" Symmetries of the Diffusion Equation

Our construction guarantees that if $f(q, t)$ is a solution to the heat equation, then $f_g(q, t)$ will also be for all values of the six free parameters a, β, γ, ξ, y, and ζ. The physical meaning of each of these parameters will be brought out now by considering the one-parameter groups.

(a) $g = \{\mathbf{1}, \mathbf{0}, i\zeta\}$: *multiplication by a constant factor.*

$$f_g(q, t) = \exp(-\zeta) f(q, t). \tag{10.18}$$

As the heat equation is *linear*, if $f(q, t)$ is a solution, any multiple of this function will be a solution as well.

(b) $g = \{\mathbf{1}, (0, y), 0\}$: *space translation* [Fig. 10.2(A)].

$$f_g(q, t) = f(q + y, t). \tag{10.19}$$

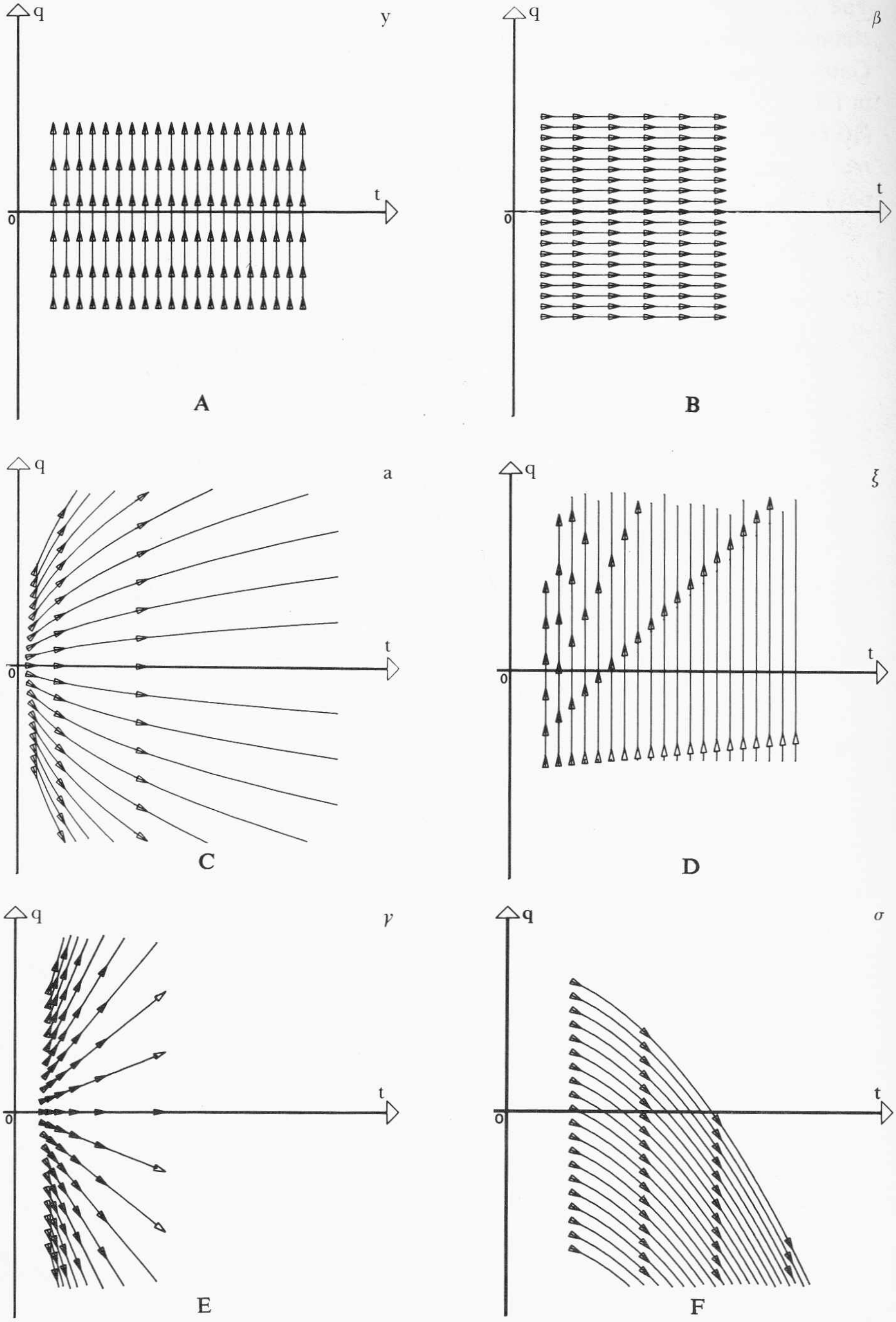


Fig. 10.2 (*see next page for legend*).

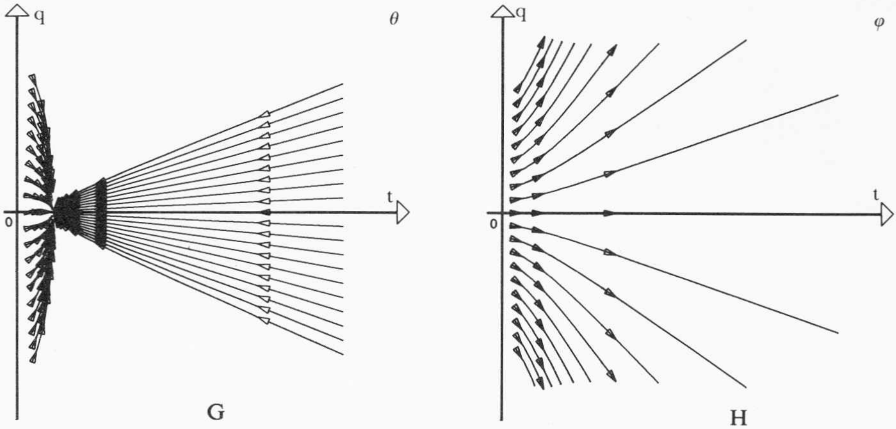


Fig. 10.2. Space and time transformation under various one-parameter subgroups of I, the similarity group of the diffusion equation. On opposite page: (A) space translation, (B) time translation, (C) scale change, (D) Galilean transformation, (E) projective transformation, (F) "linear" transformation. Above: (G) elliptic subgroup, (H) hyperbolic subgroup. Arrowheads are placed at equal intervals of the relevant parameters $y, \beta, a, \ldots$.

The heat equation (10.11) involves only derivatives with respect to q, thus describing a *homogeneous* medium: one where the medium properties are invariant under translation.

(c) $g = \left\{\begin{pmatrix} 1 & -2i\beta \\ 0 & 1 \end{pmatrix}, \mathbf{0}, 0\right\}$: *time translation* [Fig. 10.2(B)],

$$f_g(q, t) = f(q, t + \beta) \tag{10.20}$$

is due to the invariance of (10.11) under time translations. [Actually, we must restrict $\beta \geqslant 0$, as we shall see below.]

(d) $g = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \mathbf{0}, 0\right\}$: *scale change* [Fig. 10.2(C)],

$$f_g(q, t) = a^{-1/2} f(q/a, t/a^2), \tag{10.21}$$

which states basically that a change of space scale by a must be accompanied by a corresponding change of time scale by a^2.

The four transformations (10.18)–(10.21) have their origin in corresponding symmetries of the heat equation (10.11) which are "inspectionally" obvious. Our general development, however, points out *two more* transformation symmetries of the diffusion equation.

(e) $g = \{\mathbf{1}, (i\xi/2, 0), 0\}$: *Galilean transformations* [Fig. 10.2(D)].

$$f_g(q, t) = \exp[-\xi(q - \xi t)/2 - \xi^2 t/4]f(q - \xi t, t). \tag{10.22}$$

This transformation is called "Galilean" as it relates the temperature distribution in a fixed rod $f(q, t)$ to that of a rod *moving* with velocity ξ. For f_g to be a solution, f needs to be corrected by a multiplier function (10.17a)–(10.17d)–(10.17f). Figure 10.2(D) only shows the space–time transformation.

(f) $g = \left\{\begin{pmatrix} 1 & 0 \\ i\gamma/2 & 1 \end{pmatrix}, \mathbf{0}, 0\right\}$: *projective transformations* [Fig. 10.2(E)].

$$f_g(q, t) = (1 + \gamma t)^{-1/2} \exp[-\gamma q^2/4(1 + \gamma t)]f\big(q/(1 + \gamma t), t/(1 + \gamma t)\big). \tag{10.23}$$

These transformations deform the (q, t)-space in such a way that the q/t-constant lines are mapped onto themselves [see Fig. 10.2(E)].

The set of all transformations (10.17)–(10.23) constitutes the *similarity group* of the heat equation (10.11). This group, we have seen, is the inhomogeneous, real linear transform group *I*. Four one-parameter groups are "inspectional" and two are "hidden." Equations (10.17) include all of the transformations at once, so other one-parameter groups can be analyzed.

(g) $g = \left\{\begin{pmatrix} 1 & -2i\sigma \\ 0 & 1 \end{pmatrix}, (2i\sigma, 2\sigma^2, 2i\sigma^3/3)\right\}$ from Eq. (10.9), *linear subgroup* [Fig. 10.2(F)].

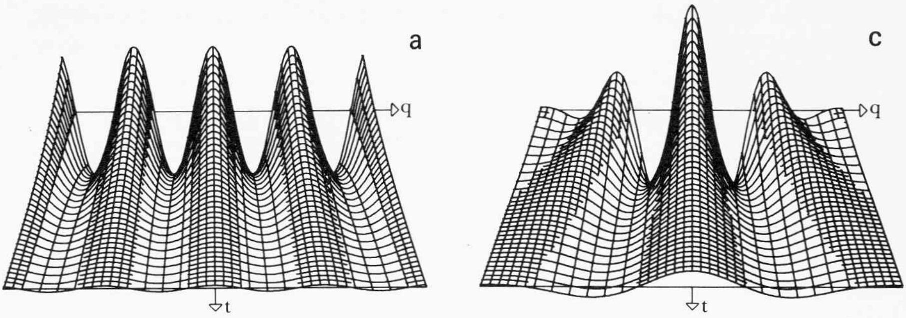
$$f_g(q, t) = \exp(-2q\sigma + 4t\sigma^2 + 4\sigma^3/3)f(q - 4t\sigma - 2\sigma^2, t + \sigma). \tag{10.24}$$

(h) $g = \{\mathbf{M}(2i\theta), \mathbf{0}, 0\}$, $\mathbf{M}(2i\theta)$ from Eq. (9.31), *elliptic subgroup* [Fig. 10.2(G)]. The transformations of space–time are

$$\begin{aligned} q_g &= q/(\cosh\theta + 2t\sinh\theta), \\ t_g &= (t\cosh\theta + \tfrac{1}{2}\sinh\theta)/(\cosh\theta + 2t\sinh\theta). \end{aligned} \tag{10.25}$$

(i) $g = \{\mathbf{M}(2i\phi), \mathbf{0}, 0\}$, $\mathbf{M}(2i\phi)$ from Eq. (9.30), *hyperbolic subgroup* [Fig. 10.2(H)]. The space–time transformations are

$$\begin{aligned} q_g &= q/(\cos\phi - 2t\sin\phi), \\ t_g &= (t\cos\phi + \tfrac{1}{2}\sin\phi)/(\cos\phi - 2t\sin\phi). \end{aligned} \tag{10.26}$$



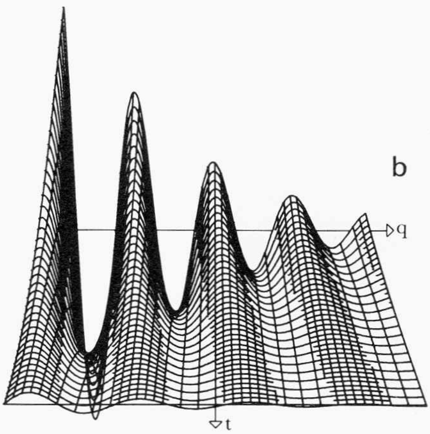


Fig. 10.3. The action of various similarity transformations on a solution of the diffusion equation. (a) A "normal mode" subject to (b) Galilean and (c) projective transformation. The original function changes due to its product with a multiplier and the transformation of space and time arguments. Regions where the plotted function is positive are drawn with a closer grid. Note that the zeros of the function move along $q =$ constant lines in (a), on $q - \xi t =$ constant lines in (b), and on $q/(1 + \gamma t) =$ constant lines in (c).

In the last two cases the appropriate multiplier function can be found from Eqs. (10.17).

The multiplier action of I is a deformation both of space–time and of the solution function. We show this in Fig. 10.3. We present one particular (*separable*) solution in Fig. 10.3(a). After acting on this $f(q, t)$ with a *Galilean* transformation, we obtain Fig. 10.3(b). A projective transformation gives the solution shown in Fig. 10.3(c).

Exercise 10.5. Verify that the set of geometric transforms (10.15) is a five-parameter subgroup of I. What is its effect on the $t = 0$ line?

Exercise 10.6. Verify that (a)–(i) are indeed subgroups of I, in particular, check in the cases of Galilean, projective, and "linear" transformations that under the product of two transformations the coordinates and multipliers compose properly. Note that for a two-variable-function $f(q, t)$ solution of (10.11), all of

I is what we could call a *geometric* transformation group in analogy with (10.15). Find its generators as two-variable first-order differential operators. [See Miller (1977, Section 2.2).]

Exercise 10.7. Verify, especially for Galilean, projective, and linear transformations, that $f_g(q, t)$ is a solution to the diffusion equation (10.11) if $f(q, t)$ is.

Exercise 10.8. Show that, as suggested by Fig. 10.2, the following, considered as sets, are the *invariant contours* under each subgroup: (a) points (q, t); (b) lines $t =$ constant; (c) lines $q =$ constant; (d) convertical parabolas; (e) lines $t =$ constant; (f) concurrent lines; (g) parallel parabolas; (h) concurrent, concentric ellipses and hyperbolas; (i) concentric, nested hyperbolas. Note that all intercepts with the $t = 0$ axis have zero slope in cases (g)–(i).

Exercise 10.9. Not all solutions of the diffusion equation can be meaningfully regressed in time. (Recall Exercise 5.3 and the discussion of the inversion of the Gauss–Weierstrass transform in Section 9.2.) We should thus demand that all time translations (10.20) be nonnegative ($\beta \geqslant 0$), so that the $t = 0$ line remains uncrossed when space–time transformations are applied. Show that consistency then requires that scale and conformal transformations also be nonnegative ($a > 0$ and $\gamma \geqslant 0$) in (10.21) and (10.23). This defines a *subsemigroup* of I. The projective and elliptic transformations, in any case, are not globally continuous. As long as we remain in the vicinity of the $t = 0$ line and use small transformation parameters, however, these features will not bother us.

The solution of differential equations with boundary conditions was the motivation for Sophus Lie to develop his work on continuous groups at the end of the last century. More recently Ovsjannikov (1962) and Bluman and Cole (1974) have updated Lie's work and spurred renewed interest in *similarity* methods. The fact that our I is the similarity group of the diffusion equation was rediscovered by Bluman and Cole (1969) and applied to a variety of boundary conditions [see, for instance, Bluman (1974)]. Another approach to the problem has been to determine all coordinate systems where the diffusion equation (10.11) *separates* into two ordinary differential equations. This has been the theme for a series of papers by Miller, Kalnins, and Boyer. The diffusion equation is specifically analyzed in the article by Kalnins and Miller (1974). A very thorough account of the method and a complete reference list can be found in the recent book by Miller (1977). Canonical transforms, as presented here (Wolf, 1976), allow us to analyze some of these problems for the parabolic differential equations (10.1)–(10.2), reducing them, as we have seen in the sample development (10.13)–(10.17), to 2×2 matrix algebra.

10.1.5. Similarity Solutions, Separation of Variables, and *R*-Separability

We shall now tackle the problem of separation of variables for the diffusion equation. The completeness of the solution and its extension to other parabolic equations will be shown in Section 10.2. Here we shall suggest the connection between separation of variables and the *similarity solutions* of the diffusion equation, that is, those solutions which at time $t = 0$ are the eigenfunctions of differential operators (10.1).

Consider the time development of the oscillating eigenfunctions $\Psi^f_{\lambda,\sigma}(q)$ of $\mathbb{H}^f$ [Eqs. (9.76b) and (9.87a)] under diffusion:

$$\Psi^{f,H}_{\lambda,\sigma}(q, t) := (\mathbb{I}_{H(t)}\boldsymbol{\Psi}^f_{\lambda,\sigma})(q) = \left[\mathbb{C}\begin{pmatrix}1 & -2it\\ 0 & 1\end{pmatrix}\boldsymbol{\Psi}^f_{\lambda,\sigma}\right](q)$$

$$= \exp[i(2it)\mathbb{H}^f]\Psi^f_{\lambda,\sigma}(q) = \exp(-2\lambda t)\Psi^f_{\lambda,\sigma}(q). \qquad (10.27)$$

The key has been the self-reproducing property (9.86) of the eigenfunctions $\Psi^f_{\lambda,\sigma}(q)$ under canonical transforms generated by its operator $\mathbb{H}^f$. See Fig. 10.3(a). Simple as it seems, (10.27) contains the interesting information that $\Psi^{f,H}_{\lambda,\sigma}(q, t)$ is a *separable* function of t and q. In these coordinates the diffusion equation manifestly *separates*, i.e., if we assume the solution has the structure $T(t)Q(q)$ and place this form in (10.11), we obtain two ordinary differential equations for the factors linked by a separation constant which we relate to λ. The solutions for $Q_\lambda(q)$ and $T_\lambda(t)$ yield precisely (10.27). Somewhat redundantly, note that under time translations (10.20), Fig. 10.2(B) draws out the $q =$ constant lines which serve as part of the grid of this coordinate system; vertical $t =$ constant lines are not drawn but can easily be added.

Exercise 10.10. Show that the diffusion equation (10.11) has separable solutions in the coordinates $q' = q - \xi t$, $t' = t$. Verify that the solutions thus obtained by the usual separation of variables method agree. What about (10.23), (10.24), (10.25), and (10.26)? The answer, *R-separation*, is analyzed below.

Now consider the nontrivial problem of exploring the diffusive time development of an initial temperature distribution of the shape of an Airy function (Fig. B.3). More precisely, consider $\Psi_\lambda{}^l(q)$ as given by Eq. (8.88), an eigenfunction of $\mathbb{H}^l$ which is the generator of the *linear* subgroup (10.9). Retracing our steps in (10.27), using the composition formula (10.8)–(10.9) for $\tau = 2it$ and the geometric action (10.15), we proceed as follows:

$$\begin{aligned}
\Psi^{l,H}_\lambda(q, t) &:= (\mathbb{I}_{H(t)}\boldsymbol{\Psi}_\lambda{}^l)(q)\\
&= \left[\mathbb{C}\begin{pmatrix}1 & -2it\\ 0 & 1\end{pmatrix}\boldsymbol{\Psi}_\lambda{}^l\right](q)\\
&= \Bigg[\mathbb{I}\{\mathbf{1}, (-2it, 2t^2, -2it^3/3)\}\\
&\qquad \times \mathbb{I}\left\{\begin{pmatrix}1 & -2it\\ 0 & 1\end{pmatrix}, (2it, 2t^2, -2it^3/3)\right\}\boldsymbol{\Psi}_\lambda{}^l\Bigg](q)\\
&= \exp(-2\lambda t)[\mathbb{I}\{\mathbf{1}, (-2it, 2t^2, -2it^3/3)\}\boldsymbol{\Psi}_\lambda{}^l](q)\\
&= \exp(2tq + 8t^3/3 - 2\lambda t)\Psi_\lambda{}^l(q + 2t^2)\\
&= \exp(2tv)\exp(-2\lambda t - 4t^3/3)\Psi_\lambda{}^l(v), \qquad v := q + 2t^2. \qquad (10.28)
\end{aligned}$$

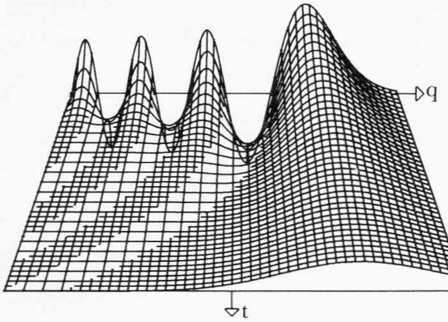


Fig. 10.4. Time development of a linear potential (Airy) $\Psi_\lambda{}^l(q)$ wave function under the diffusion equation. Regions where the function is positive are drawn with a closer grid. Notice that the zeros lie on parallel parabolas.

This function is shown in Fig. 10.4. The same remarks we made about (10.27) are valid here—with some qualification: $\Psi_\lambda^{l,H}(q, t)$ is an *R-separable* function of $v := q + 2t^2$ and t. By R-separability (or *separability with a modulation factor*) we mean a family of functions depending on a parameter λ such that

$$\Phi_\lambda(q, t) = R(u, v)U_\lambda(u)V_\lambda(v), \qquad u = u(q, t), v = v(q, t). \tag{10.29}$$

[See Morse and Feshbach (1953, p. 518) and Kalnins and Miller (1974, and the references therein).] This enlarged separability definition can be seen to work for the problem at hand. We substitute $u := t$ and $v := q + 2t^2$ into the diffusion equation (10.11), using $\partial/\partial q = \partial/\partial v$, $\partial/\partial t = \partial/\partial u + 4u\, \partial/\partial v$ and the assumed solution form (10.29). We divide by Φ_λ, suppressing arguments for brevity and indicating partial derivatives by subindices and total ones by primes. Rearranging terms slightly, we find

$$\frac{V''}{V} + \frac{V'}{V}\left(2\frac{R_v}{R} - 4u\right) - \frac{R_u}{R} = -\frac{R_{vv}}{R} + \frac{U'}{U} + 4u\frac{R_v}{R}. \tag{10.30}$$

If this equation is to separate, one member being only a function of v and the other of u, we expect all terms in R to cancel out and the expression in parentheses to be only a function of v, say, $\varphi(v)$. The solution to $R_v = [2u + \varphi(v)]R$ is $R = \chi(v)\exp(2uv)$ or, since any function of v alone will be absorbed into $V(v)$, $R = c\exp(2uv)$ with c constant. Substituting this result into (10.30), we find

$$\frac{V''}{V} - 2v = \frac{U'}{U} + 4u^2 = k, \tag{10.31}$$

where, by the usual separation argument, k must be a constant. From (10.31) we find that $U = c'\exp(ku - 4u^3/3)$ and V must be the solution of (8.87) for $k = -2\lambda$. Multiplying these results and replacing q and t, we find precisely (10.28) up to a multiplicative constant. Of course, had we not chosen the correct $u(q, t)$ and $v(q, t)$, the terms in $R(u, v)$ would not have canceled cross-variable dependencies. This method can be used in order to *find* R-separating variables [see Kalnins and Miller (1974)]. In any case, the solution of a partial differential equation is reduced to the solution of three ordinary differential equations. It is the place to point out here that time and again we have simplified partial differential equations into ordinary ones by the device of "uncoupling," i.e., of finding a convenient basis where the Laplacian operator acts by multiplication on the expansion basis functions. This is in essence the transform method. Here, we are simply expanding a function into a similar basis set, eigenfunctions of an operator which is *not* necessarily the one appearing in the differential equation. What we are using are operators which generate various transformation families within a *group* of integral transforms. The useful feature is that a group is binding together the two different operators and their eigenbases.

The separating variables obtained here can be seen in Fig. 10.2(F), where the q-t transformations generated by the *separating* operator $\mathbb{H}^l$ [i.e., Eq. (10.9)] leave the parallel parabolas $q + 2t^2 = v =$ constant invariant. The $u =$ constant lines are vertical. As we did with the q-t-separable functions before, we can subject the initial condition of (10.28) to various kinds of canonical transforms whose effect will be to produce a continuum of other separating variable pairs which will turn the above parabolas into similar conics. As this continuum can be obtained from the t, $v(q, t)$ pair by canonical transformations on the initial conditions, we shall call them all *equivalent*. Note, however, that $t, v = q + 2t^2$, and t, q are *not* equivalent but quite distinct. In Section 10.2 we shall further clarify this notion of equivalence.

We can now use $\mathbb{H}^r$ and $\mathbb{H}^h$ [Eqs. (9.76d)–(9.76e) and (9.77d)–(9.77e)] as separating operators. To this end, we consider the time development of their eigenfunctions $\Psi^r_{\lambda,\mu}(q)$ and $\Psi_n{}^h(q)$ [Eqs. (9.87b)–(9.87c)]. Repeating the basic scheme of Eq. (10.28), we obtain

$$\begin{aligned}\Psi^{r,H}_{\lambda,\sigma}(q, t) &:= (\mathbb{I}_{H(t)}\boldsymbol{\Psi}^r_{\lambda,\sigma})(q)\\ &= \exp(-\lambda \arctan 2t)\\ &\quad\times \left[\mathbb{I}\left\{\begin{pmatrix}(1+4t^2)^{1/2} & 0\\ 2it(1+4t^2)^{-1/2} & (1+4t^2)^{-1/2}\end{pmatrix}, (0,0,0)\right\}\boldsymbol{\Psi}^r_{\lambda,\sigma}\right](q)\\ &= \exp[-tq^2/(1+4t^2)](1+4t^2)^{-1/4}\\ &\quad\times \exp(-\lambda \arctan 2t)\Psi^r_{\lambda,\sigma}(q(1+4t^2)^{-1/2}),\end{aligned} \tag{10.32}$$

$$\begin{aligned}\Psi^{h,H}_n(q, t) &:= (\mathbb{I}_{H(t)}\boldsymbol{\Psi}_n{}^h)(q)\\ &= \exp[-(n+1/2)\operatorname{arctanh} 2t]\\ &\quad\times \left[\mathbb{I}\left\{\begin{pmatrix}(1-4t^2)^{1/2} & 0\\ -2it(1-4t^2)^{-1/2} & (1-4t^2)^{-1/2}\end{pmatrix}, (0,0,0)\right\}\boldsymbol{\Psi}_n{}^h\right](q)\\ &= \exp[tq^2/(1-4t^2)](1-4t^2)^{-1/4}\\ &\quad\times \exp[-(n+1/2)\operatorname{arctanh} 2t]\Psi_n{}^h(q(1-4t^2)^{-1/2}).\end{aligned} \tag{10.33}$$

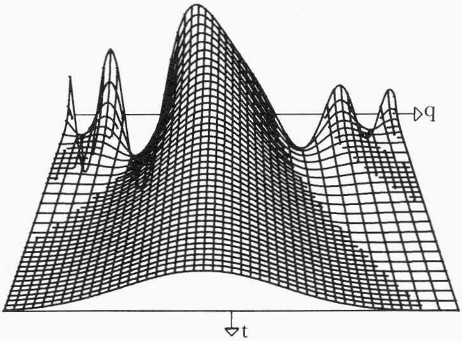


Fig. 10.5. Time development of the repulsive oscillator wave function $\Psi^r_{0,+}(q)$ under the diffusion equation. Again, the closer grid marks the regions where the function is positive. The zeros lie on concentric hyperbolas.

These are shown in Figs. 10.5 and 10.6. As in the former case, these function families are R-separable as in (10.29) for $v := q(1 + 4t^2)^{-1/2}$ and $v := q(1 - 4t^2)^{-1/2}$, respectively, and $u := t$. These can be seen in Figs. 10.2(H) and (G). These hyperbolas and ellipses are invariant under transformation generated by $\mathbb{H}^r$ and $\mathbb{H}^h$, respectively. Note that all separating coordinates we have found are such that $v(q, 0) = q$. If we apply the action of the similarity group to the (q, t)-separated normal modes (10.27), we find in general R-separated solutions. Thus, Galilean and projective transformations are R-separated in $(q - \xi t, t)$ and $[q/(1 + \gamma t), t]$, respectively. See Figs. 10.3(d) and (e).

Exercise 10.11. Follow the proof of (10.32) and (10.33) with care. Show that the diffusion equation indeed R-separates as suggested.

Exercise 10.12. In spite of the apparent singularity of (10.33) for $t = 1/2$, show that the solutions extend unscathed beyond this time.

10.1.6. The Heat Polynomials

Exercises 10.13–10.15 introduce certain solutions to the diffusion equation termed *heat polynomials*.

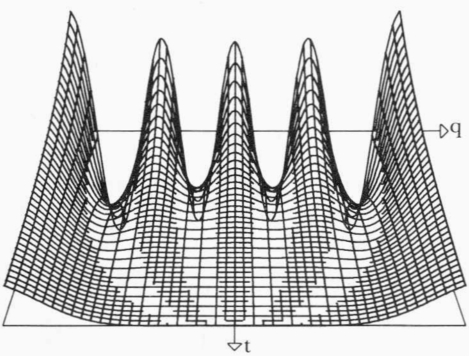


Fig. 10.6. Time development of the harmonic oscillator wave function $\Psi^h_8(q)$ under the diffusion equation. The zeros lie on concentric, convertical ellipses.

Exercise 10.13. Consider the following one-parameter subgroup of canonical transforms:

$$\exp\left[i\tau\left(-\frac{1}{2}\frac{d^2}{dq^2} + q\frac{d}{dq} + \frac{1}{2}\right)\right] = \mathbb{I}\left\{\begin{pmatrix}\exp(-i\tau) & -\sin\tau \\ 0 & \exp(i\tau)\end{pmatrix}, (0, 0, 0)\right\}. \quad (10.34)$$

Prove the equality along the same lines suggested in (10.9)–(10.10), i.e., find the integral kernel, and then show that its τ derivative equals i times the action of the operator in the exponent [the analogue of (9.75)]. In case this procedure looks tedious, we promise the reader that a streamlined process to relate exponentials of arbitrary second-order operators and the matrices representing their canonical transform subgroups will be given in Section 10.2.

Exercise 10.14. The importance of (10.34) is that the eigenfunctions of the exponent operator are the *Hermite polynomials* $H_n(q)$ with eigenvalue $n + 1/2$. Show this from (7.166) and (7.170). Hermite polynomials can be obtained from $\Psi_n^h(q)$ by a $\mathbb{C}\begin{pmatrix}1 & 0 \\ -i & 1\end{pmatrix}$ transform as it multiplies functions by $\exp(q^2/2)$. Explore this complex canonical transform: in (9.43) $u = 1 = w/2$ but $v = 0$. Refer to (9.71) to show that, indeed, Hermite polynomials are orthogonal with the weight function $\exp(-x^2)$.

Exercise 10.15. Follow the time development of a temperature distribution $H_n(q)$ by the analogue of (10.27), (10.28), (10.32), and (10.33), i.e., expressing $\mathbb{I}_{H(t)}$ as the product of a geometric transform and (10.34) for $\tau = (-i/2)\ln(1 - 4t)$. The result is found to be

$$H_n(q, t) := (1 - 4t)^{n/2} H_n(q(1 - 4t)^{-1/2}) =: 2^n v_n(q, t - 1/4), \quad (10.35)$$

where $v_n(q, t')$ are the *heat polynomials.* The time evolution of these and their relation to *power* functions for $t = 1/4$ can be seen from Eq. (7.193). There is considerable literature on these [see Hartmann and Wintner (1950), Rosenbloom and Widder (1959), Widder (1962, 1975, Chapter X), and Bilodeau (1974)]. Show that the functions (10.35) have the following properties: (a) They are *polynomials* in q and t. (b) $v_n(q, 0) = q^n$. (c) They are separable functions of t and $v = q(1 - 4t)^{-1/2}$. (d) $v(q, 0) = q$. (e) The multiplier factor R is unity, i.e., we have the case of *ordinary* separation.

We have worked out the simple diffusion equation in some detail, finding new families of solutions which will be shown in Section 10.2 to constitute, up to *equivalence, all* separable solutions to the equation. The reasons for being particularly interested in such solutions are the following. We have remarked that all separating coordinates are such that $v(q, 0) = q$. If the initial temperature distribution has a number of zeros at, say, $q_0, q_1, \ldots$, then, for all subsequent times, the separated solution (10.28) will have zeros at these values of v. The zero-temperature points will thus draw out the lines in Fig. 10.2. The solutions appear as in Fig. 10.3, and similar ones produced by different choices of (appropriate) separating operators, as in Figs. 10.4–10.6. Figure 10.3(a) should bring to memory the diffusive medium between *cold* *walls*. In a manner reminiscent of the annular membrane of Section 8.3, if we are able to solve the Sturm–Liouville problem between two values, say, q_0 and q_1, of one of the separating operators, we shall be able to describe the solutions of the diffusion equation between *moving* cold walls or similar time-dependent boundary conditions. This is a "distorted image" method adapted to those boundaries which follow conics. In Section 10.2 we shall show how the more general parabolic equation (10.1)–(10.2) is subject to this treatment by nothing more than—properly applied—matrix algebra.

10.2. Inhomogeneous Linear Canonical Transforms and Parabolic Equations

In this section we shall examine the class of second-order parabolic differential equations (10.1)–(10.2) and see that the concepts developed for the diffusion equation in Section 10.1 can be set up in a general framework applicable to the whole class. We first show how all operators of interest can be reduced by *orbit* analysis of I to essentially four subclasses *represented* by $\mathbb{H}^f$, $\mathbb{H}^l$, $\mathbb{H}^r$, and $\mathbb{H}^h$ [respectively, the free-particle, linear potential, repulsive, and harmonic oscillator quantum Hamiltonians, Eqs. (9.76b), (7.61), (9.76d), and (9.76e)]. The similarity group of the whole class is I, the inhomogeneous linear canonical transformation group. This will determine the *similarity* solutions: eigenfunctions of operators in the set, separating variables, and invariant boundaries.

10.2.1. Transformation of Operators

Consider the set of operators $\mathscr{S}_1 := \{D\mathbb{Q} + E\mathbb{P} + F\mathbb{1};\, D, E, F \in \mathscr{C}\}$ introduced in Section 10.1 and its transformation under the action of $\mathbb{I}\{g\}$, Eq. (10.7), where $g = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), (x, y, z)\} \in I$, the group of real inhomogeneous linear transformations. This can be found from (9.1), (10.6), and (10.7) to be

$$\mathbb{I}\{g\}(D\mathbb{Q} + E\mathbb{P} + F\mathbb{1})\mathbb{I}\{g\}^{-1} = (dD - cE)\mathbb{Q} + (aE - bD)\mathbb{P} \\ + (F + yD + xE)\mathbb{1}, \tag{10.36}$$

that is, I transforms $\mathscr{S}_1$ onto itself. *Second-order* operators $\mathscr{S}_2 := \{\text{Eq. (10.1)};\, A, B, \ldots, F \in \mathscr{C}\}$ can also be transformed by I onto themselves. It will be most convenient to rewrite their general expression as

$$\mathbb{H} = \sum_{k=0}^{5} \theta_k \mathbb{J}_k, \tag{10.37a}$$

$$\theta_0 = 2(A + C), \qquad \theta_1 = B, \qquad \theta_2 = 2(A - C), \qquad \theta_3 = D, \qquad \theta_4 = E, \\ \theta_5 = F, \tag{10.37b}$$

where we are using (9.76e), (9.76d), and (9.76a) for $\mathbb{J}_0$, $\mathbb{J}_1$, and $\mathbb{J}_2$ and defining

$$\mathbb{J}_3 := \mathbb{Q}, \qquad \mathbb{J}_4 := \mathbb{P}, \qquad \mathbb{J}_5 := \mathbb{1}. \tag{10.37c}$$

The action of I on $\mathscr{S}_2$ can be determined from (10.36) plus a little algebra. It has the general form

$$\mathbb{H}_g := \mathbb{I}\{g\}\mathbb{H}\mathbb{I}\{g\}^{-1} = \sum_{k=0}^{5} \theta'_k \mathbb{J}_k =: \sum_{j,k=0}^{5} \Gamma_{kj}(g)\theta_j \mathbb{J}_k. \tag{10.38a}$$

The $\{\theta'_k\}_{k=0}^5$ are linear functions of the original $\{\theta_k\}_{k=0}^5$, which transform as the entries of a column vector $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_5)^T$ under a matrix $\boldsymbol{\Gamma}(g)$ which *represents* the group element $g \in I$. Explicitly,

$$\boldsymbol{\Gamma}(g) = \begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & -cd-ab & 0 & 0 & 0 \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & cd-ab & 0 & 0 & 0 \\ -bd-ac & bd-ac & ad+bc & 0 & 0 & 0 \\ \frac{1}{2}(cx+dy) & \frac{1}{2}(cx-dy) & \frac{1}{2}(-cy-dx) & d & -c & 0 \\ \frac{1}{2}(-ax-by) & \frac{1}{2}(-ax+by) & \frac{1}{2}(ay+bx) & -b & a & 0 \\ \frac{1}{4}(x^2+y^2) & \frac{1}{4}(x^2-y^2) & -\frac{1}{2}xy & y & -x & 1 \end{pmatrix} \tag{10.38b}$$

The reason for being interested in these transformations of $\mathscr{S}_2$ is that a given operator $\mathbb{H}$ may be mapped onto a simpler, known operator $\mathbb{H}_g$. If $\psi_\lambda(q)$ is an eigenfunction of $\mathbb{H}$ with eigenvalue λ, then $\psi_\lambda{}^g(q) := (\mathbb{I}\{g\}\boldsymbol{\psi}_\lambda)(q)$ will be an eigenfunction of $\mathbb{H}_g$ with the same eigenvalue. If the latter eigenfunctions are known and the transformation g is a *geometric* one [Eq. (10.15)], then the eigenfunctions of $\mathbb{H}$ can be found simply as $\psi_\lambda(q) = (\mathbb{I}\{g\}^{-1}\boldsymbol{\psi}_\lambda{}^g)(q)$.

Exercise 10.13. Prove (10.36) by considering $g = \{\mathbf{A}, \boldsymbol{\xi}, z\}$,

$$\mathbb{I}\{g\}\mathbb{I}\{\mathbf{1}, (D, E), F\}\mathbb{I}\{g\}^{-1} = \mathbb{I}\{\mathbf{1}, (D, E)\mathbf{A}^{-1}, F - (D, E)\boldsymbol{\Omega}\boldsymbol{\xi}^T\}, \tag{10.39}$$

where $\boldsymbol{\Omega} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as in (10.8). Elements $\mathbb{H}$ of $\mathscr{S}_1$ may be seen as vectors in three-dimensional space with components $(D, E, F)^T$ which transform as the entries of (10.39) under the lower-right 3×3 submatrix of (10.38b). Note in particular that the column vector $(D, E)^T \to \mathbf{A}^{-1T}(D, E)^T$.

Exercise 10.17. Consider the composition of two transformations (10.39). Show that $\mathbb{I}\{g_1\}\mathbb{I}\{g_2\} = \mathbb{I}\{g_1 g_2\}$ acting on $(D, E, F)^T$. You may come to use $\mathbf{A}\boldsymbol{\Omega}\mathbf{A}^T = \boldsymbol{\Omega}$ [this only says that $\mathbf{A}$, an $SL(2, \mathscr{R})$ matrix, is also a two-dimensional *symplectic* one].

Exercise 10.18. Extend the above considerations to $\mathbb{H}$ seen as six-dimensional column vectors $\boldsymbol{\theta}$. Show that $\boldsymbol{\Gamma}(g_1)\boldsymbol{\Gamma}(g_2) = \boldsymbol{\Gamma}(g_1 g_2)$, $\boldsymbol{\Gamma}(\{\mathbf{1}, \mathbf{0}, 0\}) = \mathbf{1}$, and $\boldsymbol{\Gamma}(g^{-1}) = \boldsymbol{\Gamma}(g)^{-1}$ and that associativity holds. The set of matrices $\{\boldsymbol{\Gamma}(g), g \in I\}$ constitutes a 6×6 *matrix representation* of I.

10.2.2. Orbit Analysis

The concrete form of $\mathbf{\Gamma}(g)$ can be used to obtain all operators $\mathbb{H}_g = \mathbb{I}\{g\}\mathbb{H}\mathbb{I}\{g\}^{-1}$, $g \in I$, which can be "reached" from a given, fixed, operator $\mathbb{H}$. We shall consider first the case of operators with *real* coefficients θ_i. We are here specifically interested in *second*-order parabolic equations (10.1), with a nonzero leading coefficient, which will generate one-parameter subgroups (or *subsemigroups*) of canonical transforms describing the time evolution of initial conditions. If we multiply an operator $\mathbb{H}$ by a real constant κ, we are effectively only changing the time scale without affecting anything essential in the system. Similarly, addition of a term $k\mathbb{1}$ to $\mathbb{H}$ in (10.37) only multiplies the solution of the equation by a factor of $\exp(ikt)$, which we deem unimportant (see, however, Exercise 10.20).

We are led thus to consider equivalence classes of operators,

$$\Omega^\omega := \{\kappa\mathbb{I}\{g\}\mathbb{H}^\omega\mathbb{I}\{g\}^{-1} + k\mathbb{1};\, g \in I;\, \kappa, k \in \mathscr{R}\}, \tag{10.40}$$

which we shall call *the orbit of* $\mathbb{H}^\omega$. Transformation by I, multiplication by κ, and addition of a constant define an equivalence relation (see Exercise 10.19) which will *divide* $\mathscr{S}_2$ *into disjoint orbits.*

Exercise 10.19. Show that (10.40) are indeed *equivalence* classes; that is, the defining equivalence relation ($\equiv$) is (a) reflexive, $\mathbb{H} \equiv \mathbb{H}$; (b) symmetric, $\mathbb{H} \equiv \mathbb{H}' \Leftrightarrow \mathbb{H}' \equiv \mathbb{H}$; and (c) transitive, $\mathbb{H} \equiv \mathbb{H}'$, $\mathbb{H}' \equiv \mathbb{H}'' \Rightarrow \mathbb{H} \equiv \mathbb{H}''$. The relation ($\equiv$) thus divides $\mathscr{S}_2$ into disjoint sets.

We shall now show that there are exactly *six* orbits (10.40) in $\mathscr{S}_2$ of which two are trivial (one is in $\mathscr{S}_1$ and the other is the orbit of the zero operator). For the four remaining orbits we shall choose $\mathbb{H}^f$, $\mathbb{H}^l$, $\mathbb{H}^r$, and $\mathbb{H}^h$ as representatives.

The orbit analysis of $\mathscr{S}_2$ is aided by constructing

$$\Theta := \theta_0{}^2 - \theta_1{}^2 - \theta_2{}^2, \tag{10.41}$$

associated to the vector $\boldsymbol{\theta}$ which characterizes a given $\mathbb{H} \in \mathscr{S}_2$. It is straightforward to verify that this number is *invariant* under all transformations (10.38). [In group language, the subspace of $\mathscr{S}_2$ with $\theta_3 = \theta_4 = \theta_5 = 0$ is isomorphic to a three-dimensional Minkowski space–time under $SL(2, \mathscr{R}) \simeq SO(2, 1)$ transformations.] By multiplying $\mathbb{H}$ by the constant κ, Θ is multiplied by κ^2; this cannot change its *sign*, and hence we know that there are at least three orbits in $\mathscr{S}_2$ corresponding to $\Theta > 0$, $\Theta < 0$, and $\Theta = 0$. Examples of operators in these orbits are $\mathbb{H}^h$ (with $\theta_0 = 2$) for which $\Theta = 4$; $\mathbb{H}^r$ ($\theta_1 = 2$), $\Theta = -4$; and $\mathbb{H}^l$ and $\mathbb{H}^f$ ($\theta_0 = 1 = \theta_1$), $\Theta = 0$. We shall now examine these three cases and see whether we can find transformations g which map arbitrary operators with these values of Θ onto the four chosen operators, which will then serve as orbit representatives. We need consider only *geometric* transformations (10.15).

(a) $\theta^2 := \Theta > 0$. For $\mathbb{H}$ given by (10.37), $\mathbb{H}^h = 2\theta^{-1}\mathbb{I}\{g\}\mathbb{H}\mathbb{I}\{g\}^{-1} + k\mathbb{1}$, with

$$a = [|\theta|/(\theta_0 + \theta_1)]^{1/2}, \qquad c = \theta_2[|\theta|(\theta_0 + \theta_1)]^{-1/2},$$
$$x = 2[\theta_4(\theta_0 - \theta_1) - \theta_3\theta_2]/\Theta, \tag{10.42}$$
$$y = 2[\theta_4\theta_2 - \theta_3(\theta_0 + \theta_1)]/\Theta.$$

We can assume that $\theta_0 + \theta_1 \neq 0$, as otherwise the operator $\mathbb{H}$ would contain no second derivative. These operators are not physically interesting and yield to simpler methods.

(b) $-\theta^2 := \Theta < 0$, $\mathbb{H}^r = 2\theta^{-1}\mathbb{I}\{g\}\mathbb{H}\mathbb{I}\{g\}^{-1} + k\mathbb{1}$, with g given again by (10.42).

(c) $\Theta = 0$. Assume first that θ_0, θ_1, and θ_2 are not all identically zero. Then $\mathbb{I}\{g\}\mathbb{H}\mathbb{I}\{g\}^{-1}$, by a free parameter μ and

$$a = [2\mu/(\theta_0 + \theta_1)]^{1/2}, \qquad c = [(\theta_0 - \theta_1)/2\mu]^{1/2}, \tag{10.43a}$$

can be brought to a form where $\theta_0' = \mu = \theta_1'$, $\theta_2' = 0$, i.e., an operator $\mu\mathbb{P}^2/2$ plus terms *linear* in $\mathbb{Q}$ and $\mathbb{P}$. A further choice of x and y such that

$$\theta_4' = -\tfrac{1}{2}a[x(\theta_0 + \theta_1) - y\theta_2 - 2\theta_4] = 0 \tag{10.43b}$$

will eliminate all first derivatives from $\mathbb{H}$. The value of the coefficient θ_3' of $\mathbb{Q}$ is then fixed, determined only by the free parameter μ as

$$\theta_3' = (2\mu)^{-1/2}[\theta_3(\theta_0 + \theta_1)^{1/2} - \theta_4(\theta_0 - \theta_1)^{1/2}]. \tag{10.43c}$$

The operator we have is thus $\mathbb{H}' = \mu\mathbb{P}^2/2 + \theta_3'\mathbb{Q} + k\mathbb{1}$. We *cannot* make θ_3' vanish, however, *unless* the expression in brackets in (10.43c) is zero to start with. The case $\Theta = 0$ therefore contains at least two subcases:

(c1) When θ_3' in (10.43c) is nonzero, we can fix μ so that $\theta_3'/\mu = 1$, thereby bringing the operator to $\mu\mathbb{H}^l$.

(c2) When $\theta_2\theta_3 - (\theta_0 - \theta_1)\theta_4 = 0$, θ_3' in Eq. (10.43c) is zero, and a choice of $\mu = 1$ transforms the operator $\mathbb{H}$ to $\mathbb{H}^f$.

There are two more orbits in the $\theta = 0$ case:

(c3) When $\theta_0 = \theta_1 = \theta_2 = 0$ but the operator is nonzero, it belongs to $\mathscr{S}_1$. It can be always "rotated" to become $\kappa\mathbb{P}$ + constant $\mathbb{1}$.

(c4) All θ's are zero. A representative of this equivalence class is $\mathbb{1}$. We shall not consider the last two subcases in what follows.

We asked for the operators $\mathbb{H}$ in (10.37) to have real coefficients. When this condition is relaxed, Θ can be a complex number which under real I

transformations is still invariant. We now have in infinity of orbits (10.40), one for each phase of Θ, and for $\Theta = 0$ a similar unfolding. Our interest here, however, centers on real Θ's: Schrödinger-type equations where the coefficients are real and diffusive-type equations where the $\mathbb{P}^2$ term is pure imaginary. An example of the latter is, of course, the diffusion equation of Section 10.1. Another example, a *Fokker–Planck* equation, will be given in Exercises 10.20 and 10.22. For diffusive equations it seems best to place the i on the time variable and use real I transformations, as before, to relate the $\mathbb{H}$ operator to a simpler operator in the same orbit.

Now, if we allow the parameters in $g \in I$ to become complex, all $\Theta \neq 0$ orbits coalesce. A change of phase $q \to q \exp(-i\pi/4)$ will turn $\mathbb{J}_0$ into $i\mathbb{J}_1$, for example. For quantum-mechanical Schrödinger equations, complex transformations are generally meaningless as the $\mathscr{L}^2(\mathscr{R})$ norm is changed, although this may be just what one needs in order to describe *decay*. For diffusion equations, the requirement that the coordinates and function remain real usually restricts the useful transformations to a real parameter subset.

To illustrate the possibilities at hand we propose the following example.

Exercise 10.20. Consider the *Fokker–Planck*-type of differential equation,

$$\frac{\partial^2}{\partial q^2} f(q, t) + \frac{\partial}{\partial q}[q f(q, t)] = \frac{\partial}{\partial t} f(q, t), \tag{10.44a}$$

which can be written as

$$\mathbb{H}^{\mathrm{FP}} f(q, t) := (2\mathbb{J}_0 + 2\mathbb{J}_1 - 2i\mathbb{J}_2 - \tfrac{1}{2}\mathbb{1}) f(q, t) = -i\partial f(q, t)/\partial(it), \tag{10.44b}$$

with $\Theta = 4$. It is thus in the same orbit as the harmonic oscillator. Verify that, for

$$g_0 := \left\{2^{-1/2}\begin{pmatrix} 1 & 0 \\ -i & 2 \end{pmatrix}, 0, 0\right\}, \qquad \mathbb{I}\{g_0\}\mathbb{H}^{\mathrm{FP}}\mathbb{I}\{g_0\}^{-1} = \mathbb{H}^h - \tfrac{1}{2}\mathbb{1}. \tag{10.45a}$$

The "normal mode" solutions for (10.44) will thus be

$$\begin{aligned} \Psi_n^{\mathrm{FP},h}(q, t) &:= \exp(t/2)\exp[i(n + 1/2)(it)](\mathbb{I}\{g_0\}^{-1}\Psi_n{}^h)(q) \\ &= 2^{1/4}\exp(-nt)\exp(-q^2/4)\Psi_n{}^h(2^{-1/2}q) \\ &= k\exp(-nt)\exp(-q^2/2)H_n(2^{-1/2}q), \end{aligned} \tag{10.45b}$$

where k is a constant. Check that (10.45b) solves (10.44a). Note that this solution is separable in q and t. Its time evolution will propagate the zeros along $q =$ constant lines, in analogy with the normal mode solutions (10.27) of the heat equation.

Exercise 10.21. Note that, in asking for a *geometric* transformation to do the job of bringing an arbitrary $\mathbb{H}$ to one of the four chosen representatives, we are leaving out three rather important operators: (a) $\mathbb{J}_2$, which generates changes of scale [Eqs. (9.77a) and (10.21)], in the same orbit as $\mathbb{H}^r = 2\mathbb{J}_1$ by (9.35); (b) $\tfrac{1}{2}\mathbb{Q}^2$, which generates projective transformations for the diffusion equation

[Eqs. (9.77c) and (10.23)], in the same orbit as $\frac{1}{2}\mathbb{P}^2$ by a Fourier $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ transform; and (c) $\mathbb{Q}$, generator of Galilean transformations (10.22) which belong to the nonzero orbit in $\mathscr{S}_1$. Show that the cases $\theta_0 = \theta_1$, specifically excluded in our treatment, can easily be incorporated by Fourier transformation.

10.2.3. Transforming Initial Conditions: The Similarity Group

To find the relation of a given operator $\mathbb{H}$ to its chosen, known, orbit representative, we can construct solutions of the former in terms of those of the latter. It is thus sufficient to treat only the orbit representative in what follows, leaving for exercises a sample calculation for the Fokker–Planck equation (10.44). Each of the operators $\mathbb{H}^\omega$ generates a one-parameter subgroup of canonical transformations

$$\mathbb{I}_{\omega(t)} := \exp(it\mathbb{H}^\omega) =: \mathbb{I}\left\{\begin{pmatrix} h_a & h_b \\ h_c & h_d \end{pmatrix}, (h_x, h_y, h_z)\right\}, \qquad h_k = h_k(t), \quad (10.46)$$

which, acting on a function $f(q)$, defines a two-variable function

$$f^\omega(q, t) := (\mathbb{I}_{\omega(t)}\mathbf{f})(q), \qquad f^\omega(q, 0) = f(q), \tag{10.47}$$

which will be a solution to the differential equation

$$\mathbb{H}^\omega f^\omega(q, t) = -i\partial f^\omega(q, t)/\partial t. \tag{10.48}$$

Now, if the initial condition $f(q)$ is subject to an I transformation $\mathbb{I}\{g\}$ and turned into another function $f_g(q)$, the subsequent time evolution of the latter—following (10.13)—will be

$$\begin{aligned} f_g{}^\omega(q, t) &:= (\mathbb{I}_{\omega(t)}\mathbf{f}_g)(q) = (\mathbb{I}_{\omega(t)}\mathbb{I}\{g\}\mathbf{f})(q) \\ &= \left[\mathbb{I}\left\{\begin{pmatrix} a_t & 0 \\ c_t & a_t^{-1} \end{pmatrix}, (x_t, y_t, z_t)\right\}\mathbb{I}_{\omega(t_g)}\mathbf{f}\right](q) = [\mathbb{I}\{G_t\}\mathbf{f}^\omega(\cdot, t_g)](q) \\ &= a_t^{-1/2}\exp[i(c_t q^2/2a_t + x_t q/a_t + x_t y_t/2 + z_t)]f^\omega(q/a_t + y_t, t_g). \end{aligned} \tag{10.49}$$

The parameters $a, c, \ldots, z$ of G_t which depend on t and the function $t_g(t)$ can be calculated from identities between the elements of the matrix representatives of $\mathbb{I}_{\omega(t)}\mathbb{I}\{g\}$ and $\mathbb{I}\{G_t\}\mathbb{I}_{\omega(t_g)}$, where, note, G_t is a *geometric* transformation. In Table 10.1 we summarize the results for the four orbit representative operators. It follows that if $f^\omega(q, t)$ is a solution to (10.48), we generate a six-parameter continuum of solutions. We thus state that the group I is the *similarity group* for the operator $\mathbb{H}^\omega$. The association seen in Section 10.1 between the various parameters and geometrical transformations such as changes of scale and Galilean and projective transformations is seen to be peculiar to the $\mathbb{H}^f$ and $\mathbb{H}^l$ cases. In general, the $v(q, t) =$ constant lines will be rather complicated curves in the (q, t)-plane. In all cases, however, we stress that I is the *full* similarity group of the differential equation (10.48) including both *manifest* and *hidden* invariances.

Exercise 10.22. Continuing with the Fokker–Planck equation introduced in Exercise 10.20, show $\mathbb{I}_{\mathrm{FP}(t)}$ to be

$$\begin{aligned}\mathbb{I}_{\mathrm{FP}(t)} &= \mathbb{I}\{g_0\}^{-1}\exp[i(it)(\mathbb{H}^h - \tfrac{1}{2}\mathbb{1})]\mathbb{I}\{g_0\}\\ &= \mathbb{I}\left\{\begin{pmatrix}\exp(-t) & -2i\sinh t\\ 0 & \exp(t)\end{pmatrix}, \mathbf{0}, -it/2\right\}\end{aligned} \tag{10.50}$$

from (10.44), (10.45), and (9.77e). Now, comparing the matrix elements of $\mathbb{I}_{\mathrm{FP}(t)}\mathbb{I}\{g\}$ and $\mathbb{I}\{G_t\}\mathbb{I}_{\mathrm{FP}(t_g)}$, show that

$$\exp(2t_g) - 1 = \{ib + d[\exp(2t) - 1]\}/\{a - ic[\exp(2t) - 1]\}, \tag{10.51a}$$

$$a_t = \exp(t_g)[a\exp(-t) - 2ic\sinh t], \tag{10.51b}$$

$$c_t = c\exp(t + t_g), \tag{10.51c}$$

$$x_t = x\exp(t_g), \tag{10.51d}$$

$$y_t = y\exp(-t_g) + 2ix\sinh t_g. \tag{10.51e}$$

In asking for the time variable and the solution to be real, we are led to consider b, c, and x pure imaginary—as in the diffusion equation of Section 10.1.

10.2.4. Similarity Solutions and Separation of Variables

In keeping with the general plan of presentation of Section 10.1, we would now like to examine the time evolution, under $\mathbb{H}^\omega$, of *similarity* solutions, i.e., eigenfunctions $\Psi_\lambda^\upsilon(q)$ of a second operator $\mathbb{H}^\upsilon$. One of the results of this development will be the definition and explicit calculation of four equivalence classes of separating coordinates for our set of differential equations. As before, let $\mathbb{I}_{\omega(t)}$ and $\mathbb{I}_{\upsilon(t)}$ be the one-parameter time-evolution subgroups generated by $\mathbb{H}^\omega$ and $\mathbb{H}^\upsilon$ [Eq. (10.46)]. Then for a finite neighborhood of $t = 0$ we can always write

$$\mathbb{I}_{\omega(t)} = \mathbb{I}\{G_t^{\omega\upsilon}\}\mathbb{I}_{\upsilon(t')}, \tag{10.52}$$

where $G_t^{\omega\upsilon}$ is a time-dependent geometric transformation binding the two evolution subgroups for $t' = t'(t)$. The parameters of $G_t^{\omega\upsilon}$ and $t'(t)$ have been collected in Table 10.2 for pairs of orbit representative operators.

Now we proceed as we did in (10.27), (10.28), (10.32), and (10.33), using the fact that $\mathbb{I}_{\upsilon(t')}$ acting on $\Psi_\lambda^\upsilon(q)$ multiplies it only by a factor of $\exp(i\lambda t')$:

$$\begin{aligned}\Psi_\lambda^{\upsilon,\omega}(q,t) &:= (\mathbb{I}_{\omega(t)}\Psi_\lambda^\upsilon)(q) = (\mathbb{I}\{G_t^{\omega\upsilon}\}\mathbb{I}_{\upsilon(t')}\Psi_\lambda^\upsilon)(q)\\ &= \exp(i\lambda t')(\mathbb{I}\{G_t^{\omega\upsilon}\}\Psi_\lambda^\upsilon)(q)\\ &= a_t^{-1/2}\exp[i(c_t q^2/2a_t + x_t q/a_t + x_t y_t/2 + z_t + \lambda t')]\\ &\quad\times \Psi_\lambda^\upsilon(q/a_t + y_t)\\ &= \exp\{i[c_t a_t v^2/2 + (x_t - c_t a_t y_t)v]a_t^{-1/2}\\ &\quad\times \exp\{i[-y_t(x_t - c_t a_t y_t)/2 + z_t + \lambda t'(t)]\}\Psi_\lambda^\upsilon(v).\end{aligned} \tag{10.53}$$

In the last expression, we have rearranged the factors so as to display the *R-separability* of the function $\Psi_\lambda^{\upsilon,\omega}(q, t)$ as the function $\Psi_\lambda{}^\upsilon(v)$ of $v(q, t) = q/a_t + y_t$ times a function of t times a *multiplier* involving v and t but independent of λ. [Recall Eq. (10.28) and the ensuing discussion.] The pair of coordinates (v, t) are the coordinates *separated by* $\mathbb{H}^\upsilon$ of the differential equation (10.48) for $\mathbb{H}^\omega$. In Table 10.3 we have collected these coordinates and the multiplier functions for all pairs of orbit representative operators.

10.2.5. Equivalent and Nonequivalent Separating Coordinates

To bring out the significance of *equivalent* coordinate systems, consider first the case when $\mathbb{H}^\omega$ and $\mathbb{H}^\upsilon$ belong to the *same* orbit, i.e., there exists a similarity group element $g \in I$ such that

$$\mathbb{H}^\upsilon = \mathbb{I}\{g\}\mathbb{H}^\omega\mathbb{I}\{g\}^{-1}, \qquad \Psi_\lambda{}^\upsilon(q) = (\mathbb{I}\{g\}\Psi_\lambda{}^\omega)(q). \tag{10.54}$$

Then

$$\begin{aligned}\Psi_\lambda^{\upsilon,\omega}(q, t) &:= (\mathbb{I}_{\omega(t)}\Psi_\lambda{}^\upsilon)(q) = (\mathbb{I}_{\omega(t)}\mathbb{I}\{g\}\Psi_\lambda{}^\omega)(q) = (\mathbb{I}\{G_t\}\mathbb{I}_{\omega(t')}\Psi_\lambda{}^\omega)(q)\\ &= \exp(i\lambda t')(\mathbb{I}\{G_t\}\Psi_\lambda{}^\omega)(q).\end{aligned} \tag{10.55}$$

Indeed, this is just (10.49) when the choice for $f(q)$ is $\Psi_\lambda{}^\omega(q)$, so that $f(q, t)$ is $\exp(i\lambda t')\Psi_\lambda{}^\omega(q)$, a separable function in q and t. The conclusion is that *when* $\mathbb{H}^\omega$ *and the separating operator* $\mathbb{H}^\upsilon$ *belong to the same orbit, the separating coordinates* (v, t) *of the former can be obtained from the Cartesian ones by a transformation in the similarity group of the equation.* These will be taken to be *equivalent*. (Compare Fig. 10.3.) Now, if $\mathbb{H}^\omega$ and $\mathbb{H}^\upsilon$ belong to *different* orbits, the separating coordinates are inequivalent.

To find all coordinate systems equivalent to a given $[v(q, t), t]$ defined by an orbit representative operator $\mathbb{H}^\upsilon$, consider the action of $\mathbb{I}_{\omega(t)}$ on $(\mathbb{I}\{g\}\Psi_\lambda{}^\upsilon)(q)$. This will give the coordinates associated to $\mathbb{I}\{g\}\mathbb{H}^\upsilon\mathbb{I}\{g\}^{-1}$. Proceeding by (10.49) and (10.53), we obtain

$$\begin{aligned}(\mathbb{I}_{\omega(t)}\mathbb{I}\{g\}\Psi_\lambda{}^\upsilon)(q) &= (\mathbb{I}\{G_t\}\mathbb{I}_{\omega(t_g)}\Psi_\lambda{}^\upsilon)(q)\\ &= (\mathbb{I}\{G_t\}\mathbb{I}\{G_{t_g}^{\omega\upsilon}\}\mathbb{I}_{\upsilon(t_g')}\Psi_\lambda{}^\upsilon)(q)\\ &= \exp(i\lambda t_g')(\mathbb{I}\{G_t G_{t_g}^{\omega\upsilon}\}\Psi_\lambda{}^\upsilon)(q).\end{aligned} \tag{10.56}$$

In other words, we have only to apply the transformations of Table 10.1, representing G_t and t_g, to those of Table 10.2, representing $G_t^{\omega\upsilon}$ and $t'(t)$. If the former are the identity, we obtain Table 10.3 from (10.53).

Exercise 10.23. Implement (10.56) for the case of the free-particle Schrödinger equation. This case leads both to algebraically manageable results and to conclusions which are relevant for the diffusion equation. Show that the coordinate systems equivalent to (q, t) (f-f box in Table 10.3) are given by

$$v = [q + (dx - cy)t + (ay - bx)]/(a - ct) \tag{10.57a}$$

and t. The coordinate systems equivalent to $(q - t^2/2, t)$ (f-l box in Table 10.3) are

$$v = [q + (dx - cy)t + (ay - bx)]/(a - ct) - \tfrac{1}{2}[(dt - b)/(a - ct)]^2 \quad (10.57b)$$

and t, and those equivalent to $\big(q(1 \mp t^2)^{-1/2}, t\big)$ (boxes f-r and f-h of Table 10.3) are

$$v = [q + (dx - cy)t + (ay - bx)][(a - ct)^2 \mp (dt - b)^2]^{-1/2} \quad (10.57c)$$

and t. Note that all the coordinate lines $v =$ constant are conic sections. Can you find their axes and foci?

Exercise 10.24. Find the multiplier exponents $S(v, t)$ which generalize those of Table 10.3 according to (10.57). Recall the transform in Exercise 10.15 which led to the heat polynomials. Show that the multiplier function which corresponds to the coordinate system separated by the generator of (10.34) is unity, as (10.35) clearly shows. This is the free-particle counterpart of the heat polynomials. Can you show that this system and the Cartesian one are essentially (up to translations and dilatations) the *only* completely ($R = 1$) separating coordinate systems?

Table 10.1 Geometric and Time Transformation Parameters for the Action of a General Element in I on the Solutions of the Differential Equation $\mathbb{H}^\omega \mathbf{f} = -i\partial_t \mathbf{f}$ [a]

Operator $\mathbb{H}^\omega$	Time $t_g(t)$ transformation	Geometrical transformation
$\mathbb{H}^f = \frac{1}{2}\mathbb{P}^2$	$t_g = \dfrac{dt - b}{a - ct}$	$a_t = a - ct,\ c_t = c,\ z_t = z,$ $(x_t, y_t) = (x, y)\begin{pmatrix}1 & t_g\\ 0 & 1\end{pmatrix}$
$\mathbb{H}^l = \frac{1}{2}\mathbb{P}^2 + \mathbb{Q}$	$t_g = \dfrac{dt - b}{a - ct}$	$a_t = a - ct,\ c_t = c,\ z_t = z,$ $(x_t, y_t) = \left[(x, y) + (t, -t^2/2)\begin{pmatrix}a & b\\ c & d\end{pmatrix} - (t_g, -t_g^2/2)\right]\begin{pmatrix}1 & t_g\\ 0 & 1\end{pmatrix}$
$\mathbb{H}^r = \frac{1}{2}(\mathbb{P}^2 - \mathbb{Q}^2)$	$\tanh t_g = \dfrac{d\tanh t - b}{a - c\tanh t}$	$a_t = (a\cosh t - c\sinh t)/\cosh t_g$ $= (d\sinh t - b\cosh t)/\sinh t_g,$ $c_t = (c\cosh t - a\sinh t + a_t^{-1}\sinh t_g)/\cosh t_g,$ $(x_t, y_t) = (x, y)\begin{pmatrix}\cosh t_g & \sinh t_g\\ \sinh t_g & \cosh t_g\end{pmatrix},$ $z_t = z$
$\mathbb{H}^h = \frac{1}{2}(\mathbb{P}^2 + \mathbb{Q}^2)$	$\tan t_g = \dfrac{d\tan t - b}{a - c\tan t}$	$a_t = (a\cos t - c\sin t)/\cos t_g$ $= (d\sin t - b\cos t)/\sin t_g,$ $c_t = (c\cos t + a\sin t - a_t^{-1}\sin t_g)/\cos t_g,$ $(x_t, y_t) = (x, y)\begin{pmatrix}\cos t_g & \sin t_g\\ -\sin t_g & \cos t_g\end{pmatrix},$ $z_t = z$

[a] The heat equation can be related to case f by substituting $t \to 2it$, $b =: -2i\beta$, $c =: i\gamma/2$, $x =: i\xi/2$, $z =: i\zeta$.

Table 10.2 Parameters and Time Transformation for the Evolution, Governed by $\mathbb{H}^{\omega}\psi_{\lambda}^{\upsilon} = -i\partial_t\psi_{\lambda}^{\upsilon}$, of the Eigenfunctions of an Operator $\mathbb{H}^{\upsilon}$ [a]

$\omega \backslash \upsilon$	f	l	r	h
f	1	$t' = t$ $a_t = 1,$ $x_t = -t$ $y_t = -t^2/2,$ $z_t = -t^3/12$	$\tanh t' = t$ $a_t = (1 - t^2)^{1/2}$ $c_t = t(1 - t^2)^{-1/2}$	$\tan t' = t$ $a_t = (1 + t^2)^{1/2}$ $c_t = -t(1 + t^2)^{-1/2}$
l	$t' = t$ $a_t = 1,$ $x_t = t$ $y_t = t^2/2,$ $z_t = t^3/12$	1	$\tanh t' = t$ $a_t = (1 - t^2)^{1/2}$ $c_t = t(1 - t^2)^{-1/2}$ $x_t = t(1 - t^2/2)(1 - t^2)^{-1/2}$ $y_t = \frac{1}{2}t^2(1 - t^2)^{-1/2}$ $z_t = -t^3/12$	$\tan t' = t$ $a_t = (1 + t^2)^{1/2}$ $c_t = -t(1 + t^2)^{-1/2}$ $x_t = t(1 + t^2/2)(1 + t^2)^{-1/2}$ $y_t = \frac{1}{2}t^2(1 + t^2)^{-1/2}$ $z_t = -t^3/12$
r	$t' = \tanh t$ $a_t = \cosh t$ $c_t = -\sinh t$	$t' = \tanh t$ $a_t = \cosh t$ $c_t = -\sinh t$ $x_t = -t'$ $y_t = -t'^2/2$ $z_t = t'^3/12$	1	$\tan t' = \tanh t$ $a_t = (\cosh 2t)^{1/2}$ $c_t = -\sinh 2t(\cosh 2t)^{-1/2}$
h	$t' = \tan t$ $a_t = \cos t$ $c_t = \sin t$	$t' = \tan t$ $a_t = \cos t$ $c_t = \sin t$ $x_t = -t'$ $y_t = -t'^2/2$ $z_t = t'^3/12$	$\tanh t' = \tan t$ $a_t = (\cos 2t)^{1/2}$ $c_t = \sin 2t(\cos 2t)^{-1/2}$	1

[a] Refer to Eqs. (10.52) – (10.53). The entry 1 means $t' = t$ and $a_t = 1$, $c_t = 0$. Missing entries are zero. (As in Table 10.1, the heat equation is related to case f by $t \to 2it$. The results appear explicitly in Section 10.1.)

If we are faced with a differential equation of the type (10.48) with given initial ($t = 0$) and moving boundary conditions, our procedure would be to see whether we can find separating variables such that the coordinate curves match the boundaries. If they do, there is an associated operator whose Sturm–Liouville problem in the appropriate interval yields the best function set in which to expand the initial $t = 0$ data, so that they naturally follow the constancy conditions at the moving boundary. This is in essence the familiar image method but applied to moving, distorting, mirrors.

Table 10.3 **Separating Coordinates $(v(q, t), t)$ and Multiplier Functions $R(v, t) = \exp[iS(v, t)]$ [Eq. (10.56)] for the Differential Equation $\mathbb{H}^{\omega}\mathbf{f} = -i\partial_t \mathbf{f}$, as Separated by the Operator $\mathbb{H}^{v}$** [a]

ω \ v	f	l	r	h
f	$v = q$ $S = 0$	$v = q - t^2/2$ $S = -vt$	$v = q(1 - t^2)^{-1/2}$ $S = v^2 t/2$	$v = q(1 + t^2)^{-1/2}$ $S = -v^2 t/2$
l	$v = q + t^2/2$ $S = vt$	$v = q$ $S = 0$	$v = (q + \frac{1}{2}t^2)(1 - t^2)^{-1/2}$ $S = v^2 t/2 + vt(1 - t^2)^{1/2}$	$v = (q + \frac{1}{2}t^2)(1 + t^2)^{-1/2}$ $S = -v^2 t/2 + vt(1 + t^2)^{1/2}$
r	$v = q/\cosh t$ $S = -v^2 \sinh(2t)/4$	$v = q/\cosh t - \frac{1}{2}\tanh^2 t$ $S = -v^2 \sinh(2t)/4$ $- v \tanh t(1 + \frac{1}{2}\sinh^2 t)$	$v = q$ $S = 0$	$v = q(\cosh 2t)^{-1/2}$ $S = -\frac{1}{2}v^2 \sinh 2t$
h	$v = q/\cos t$ $S = v^2 \sin(2t)/4$	$v = q/\cos t - \frac{1}{2}\tan^2 t$ $S = v^2 \sin(2t)/4$ $- v \tan t(1 - \frac{1}{2}\sin^2 t)$	$v = q(\cos 2t)^{-1/2}$ $S = \frac{1}{2}v^2 \sin 2t$	$v = q$ $S = 0$

[a] As before, the heat equation is related to the f case by $t \to 2it$.